Indian Statistical Institute, Bangalore
B. Math.
First Year, Second Semester
Analysis II

Final Examination | Date : May 2, 2018
Maximum marks: 100 | Time: 3 hours

1. Let $f : [0, 1] \to \mathbb{R}$ be the function $f(x) = x, \ \forall x \in [0, 1]$. From first principles, that is, through computing upper and lower sums, show that
$$\int_0^1 f = \frac{1}{2}.$$
[15]

2. Suppose $g : [0, 1] \to [0, \infty)$ is a continuous function and $\int_0^1 g = 0$. Show that $g(t) = 0$ for all $t \in [0, 1]$. [15]

3. Let Ω be a non-empty finite set and let $\mathcal{F} = \{A : A \subseteq \Omega\}$ be the power set of Ω. For $A, B \in \mathcal{F}$ define $d(A, B)$ as the number of elements in $A \triangle B := (A \bigcap B^c) \bigcup (B \bigcap A^c)$. Show that d is a metric on $\mathcal{F}$. (Hint: Use Venn diagrams). [15]

4. Let (X, d) be a compact metric space. Suppose $f : X \to \mathbb{R}$ is a continuous function. Show that $\{f(x) : x \in \mathbb{X}\}$ is compact. [15]

5. Let $h : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by
$$h(x, y) = \begin{cases} \frac{3x^4}{(x^2+y^2)} & \text{if} \quad (x, y) \neq (0, 0); \\ 0 & \text{if} \quad (x, y) = (0, 0), \end{cases}$$
Determine as to whether h is continuous at the origin or not. Compute partial derivatives $D_1 h(0, 0)$ and $D_2 h(0, 0)$ if they exist. [15]

6. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable (has total derivative) at $a \in \mathbb{R}^n$. Show that f is continuous at $a \in \mathbb{R}^n$. [15]

7. Use the method of Lagrange multipliers to find the point nearest to the origin in the plane $2x + 3y - z = 5$ in $\mathbb{R}^3$. [15]